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q-deformed ladder and shift operators for three exactly solvable potentials obeying $SO(2, 1)$ symmetry

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Abstract. The quantum deformation algebra $SO_q(2, 1)$ is studied and applied to derive the q-analogues of the ladder and shift operators for the radial Coulomb, radial harmonic oscillator and Morse oscillator potentials. The q-deformed operators in all three cases are found to act like shift operators, called *q-shift* operators. Their possible similarity with the quasi-shift operators arising in supersymmetric quantum mechanics, or factorization, of the radial harmonic oscillator is also pointed out.

1. Introduction

Recently, Cooper [1] used algebraic methods to obtain the transition operators for three exactly solvable potentials, namely the radial Coulomb, radial harmonic oscillator and Morse oscillator potentials. It is shown that each system obeys the $SO(2, 1)$ algebra‡ and the resultant transition operators act as ladder (or energy changing at constant angular momentum) operators in the cases of the radial Coulomb and radial harmonic oscillator potentials, whereas they act as shift (constant energy at different well depths) operators in the case of the Morse potential. The mappings between each pair of potentials are also exactly identified. The transition operators, more commonly known as the raising or lowering operators of the energy states, are of value in the analytical determination of the eigenvalues corresponding to exactly determined eigenstates. In this paper, we introduce the q-analogues of these transition operators in the hope of learning some new fine-structure physics, such as spectrum splitting and shift [2].

Quantum groups (or q-deformation, in short) constitute a recently introduced mathematical tool for nuclei, molecules and many other physical systems obeying statistical mechanics, conformal field theory, field theory of strings, etc. The puzzling question so far has been the absence of a universal single meaning associated with the deformation parameter q which characterizes the quantum nature of the group. To begin with, this parameter was considered to play a role equivalent to that of the Planck constant and, hence, the name 'quantum' group. However, recent work by Gupta *et al* [3] has shown that q-deformation of the Planck distribution leads from ideal black-body radiation to that of a non-ideal real body. In terms of the interacting boson model of nuclei, q is related to the softness parameter [4] of the variable moment-of-inertia model, as well as to the mixing of the dynamical symmetries [5–7] of the one-dimensional $U(2)$, two-dimensional $SU(3)$ and three-dimensional $U(6)$ groups. Gupta [6] and Gupta and collaborators [5, 7] have shown

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‡ We use the same notation for algebra and groups.

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that for complex values of the q-deformation parameter, a complete (analytical) description of the underlying group ($U(2)$, $SU(3)$ or $U(6)$) can be obtained by q-deforming only one of its limiting symmetries. This is referred to as q-breaking of dynamical symmetry [6]. The pure imaginary values of q-deformation are found [6, 8] to restore the dynamical symmetry. For molecules, not only have the one-dimensional $SU_q(2)$ and $SU_q(1, 1)$ symmetries been studied [9–11] for both rotational and vibrational spectra, but also the $O_q(4)$ limit of the three-dimensional $U(4)$ group [12]. The q-deformation parameter is shown to be related to the expansion coefficients of the empirical Dunham expansion. However, in view of the detailed work of Kim *et al* [13] on the mixing of classical (un-deformed) dynamical symmetries of $U(4)$, it seems [14] that the q-deformation parameter in $U(4)$ would also play the role of a symmetry mixing parameter, like that in $U(6)$ of nuclei [7]. One can thus elucidate the different roles of the q-deformation parameter in various different problems. This makes the study of the q-analogue of any physical problem in itself interesting, as well as important.

For the Coulomb potential, the q-analogue of the hydrogen atom energy spectrum is obtained in terms of the $SO_q(4) \sim SU_q(2) \oplus SU_q(2)$ algebra [2, 15, 16]. Also, the deformation of a four-dimensional oscillator is studied which arises in the application of the Kustaanheimo–Stiefel transformation [2]. The role of q-deformation in reproducing the 2s–2p Dirac splitting is indicated, which provides a model of the 2s–2p splitting without invoking relativistic quantum mechanics. In this paper, we study the radial Coulomb potential, as well as the other two exactly solvable potentials, which correspond to different realizations of the $SO(2, 1)$ group [17]. The transition operators of the $SO(2, 1)$ group are found to satisfy $SU(1, 1)$ algebra [1], whose q-deformation is well known [18, 19]. This allows us to construct the $SO_q(2, 1)$ algebra and, hence, the q-deformed ladder and shift operators for the three potentials considered here.

This paper is organized as follows. In section 2, we give the $SO(2, 1)$ realizations of the three potentials. The relevent ladder and shift operators are obtained in section 3 by invoking the transformation of $SO(2, 1)$ to $SU(1, 1)$. Section 4 deals with the bosonization procedure and the establishing of $SO_q(2, 1)$. The q-deformed ladder and shift operators are obtained in section 5. Finally, a summary and discussion of our results is added as section 6.

2. $SO(2, 1)$ realizations of the radial Coulomb, radial harmonic oscillator and Morse oscillator potentials

The Schrödinger equations for the three exactly solvable potentials are expressible in dimensionless form as [1]

$$\left[-\rho\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho} + \rho\right]\psi_{v,l} = 2n\psi_{v,l} \tag{1}$$

$$\left[-\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{l(l+1)}{2\xi^2} + \frac{1}{2}\xi^2\right]\psi_{v,l} = (2v + l + \tfrac{3}{2})\psi_{v,l} \tag{2}$$

$$\left[-e^x\frac{d^2}{dx^2} + (\lambda - v - \tfrac{1}{2})^2 e^x + e^{-x}\right]\psi_{v,l} = 2\lambda\psi_{v,l}. \tag{3}$$

In equation (1), $n \equiv v + l + 1$ is the principal quantum number, with v representing the number of radial nodes and, in equations (2) and (3), v represents the vibrational quantum

number. Also, l is the rotational angular momentum quantum number in equations (1) and (2) and, in equation (3), λ represents the well-depth parameter of the Morse potential which is allowed to vary only by integer amounts.

Each of these equations can be expressed in terms of three operators $W_i (i = 1, 2, 3)$, which satisfy a single common commutation relation

$$[W_1, W_3] = 2iW_2 \tag{4}$$

where the explicit forms of W_i are given by

$$\begin{aligned} W_1^{\text{Coul}} &= \rho \\ W_2^{\text{Coul}} &= 2\rho \frac{d}{d\rho} \\ W_3^{\text{Coul}} &= -\rho \frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho} \end{aligned} \tag{5}$$

for the radial Coulomb potential,

$$\begin{aligned} W_1^{\text{Harm}} &= \frac{1}{2}\xi^2 \\ W_2^{\text{Harm}} &= -\frac{i}{2}\left[\xi \frac{d}{d\xi} + \frac{1}{2}\right] \\ W_3^{\text{Harm}} &= \frac{1}{2}\left[-\frac{d^2}{d\xi^2} + \frac{l(l+1)}{\xi^2}\right] \end{aligned} \tag{6}$$

for the radial harmonic oscillator potential, and

$$\begin{aligned} W_1^{\text{Morse}} &= e^{-x} \\ W_2^{\text{Morse}} &= i\left(\frac{d}{dx} - \frac{1}{2}\right) \\ W_3^{\text{Morse}} &= -e^x \frac{d^2}{dx^2} + (\lambda - v - \tfrac{1}{2})e^x \end{aligned} \tag{7}$$

for the one-dimensional Morse potential.

Then, a new set of operators $T_i (i = 1, 2, 3)$ defined as

$$\begin{aligned} T_1 &= \tfrac{1}{2}(W_3 - W_1) \\ T_2 &= W_2 \\ T_3 &= \tfrac{1}{2}(W_3 + W_1) \end{aligned} \tag{8}$$

satisfy the commutation relations

$$\begin{aligned} [T_1, T_2] &= -iT_3 \\ [T_2, T_3] &= iT_1 \\ [T_3, T_1] &= iT_2 \end{aligned} \tag{9}$$

which characterize the $SO(2, 1)$ algebra [17]. The interesting point to note here is that the defining relations (8) of generators T_1, T_2, and T_3 of $SO(2, 1)$ algebra are the same for all three potentials. This means that all three potentials can be treated on an equal footing and a unified approach is possible.

The second-order Casimir operator of $SO(2, 1)$ is given by

$$C_2(SO(2,1)) = T_3^2 - T_1^2 - T_2^2$$
$$= W_3 W_1 - W_2^2 = W_1 W_3 - W_2(2\mathrm{i} + W_2) \tag{10}$$

which is different for different potentials. Substituting for W_i from equations (5)–(7), we get, for the three potentials, the eigenvalue equations

$$C_2^{\mathrm{Coul}} \psi_{v,l}^{\mathrm{Coul}} = l(l+1)\psi_{v,l}^{\mathrm{Coul}} \tag{11a}$$

$$C_2^{\mathrm{Harm}} \psi_{v,l}^{\mathrm{Harm}} = \tfrac{1}{4}(l(l+1) - \tfrac{3}{4})\psi_{v,l}^{\mathrm{Harm}} \tag{11b}$$

$$C_2^{\mathrm{Morse}} \psi_{v,\lambda}^{\mathrm{Morse}} = (\lambda - v)(\lambda - v - 1)\psi_{v,\lambda}^{\mathrm{Morse}} \equiv (-E_v - \tfrac{1}{4})\psi_{v,\lambda}^{\mathrm{Morse}}. \tag{11c}$$

Thus, the irreducible representations for both the radial Coulomb and radial harmonic oscillator potentials are characterized by the angular momemtum quantum number l, whereas that of the Morse potential refers to constant energy E_v.

Also, the operator $T_3 = (\frac{1}{2}(W_3 + W_1))$, which corresponds to the Casimir of the subalgebra $SO(2)$ of $SO(2, 1)$, yields an eigenvalue equation for each potential:

$$T_3^{\mathrm{Coul}} \psi_{v,l}^{\mathrm{Coul}} = n\psi_{v,l}^{\mathrm{Coul}} \equiv (v + l + 1)\psi_{v,l}^{\mathrm{Coul}} \tag{12a}$$

$$T_3^{\mathrm{Harm}} \psi_{v,l}^{\mathrm{Harm}} = \tfrac{1}{2}(2v + l + \tfrac{3}{2})\psi_{v,l}^{\mathrm{Harm}} \tag{12b}$$

$$T_3^{\mathrm{Morse}} \psi_{v,\lambda}^{\mathrm{Morse}} = \lambda \psi_{v,\lambda}^{\mathrm{Morse}}. \tag{12c}$$

It is interesting to note that for both the Coulomb and harmonic oscillator potentials, the operator T_3 determines the energy (since n in the Coulomb potential is the principal quantum number, it determines the energy), whereas for the Morse potential it represents the well-depth parameter λ. We shall see in the following section that this difference in the behaviour of the T_3 operators has an important consequence for the nature of transition operators for the radial Coulomb and radial harmonic oscillator potentials and that of the Morse potential. In the former two cases, we obtain ladder operators, whereas in the later case, we obtain shift operators.

3. The classical ladder/shift operators

The transition operators are constructed as follows:

$$T_\pm \equiv T_1 \pm \mathrm{i}T_2 = \tfrac{1}{2}(W_3 - W_1) \pm \mathrm{i}W_2 \tag{13a}$$

$$T_3 \equiv T_3 = \tfrac{1}{2}(W_3 + W_1) \tag{13b}$$

whose explicit forms for the three potentials can be obtained by substitution of W_i from equations (5)–(7). The Casimir operator (10) then takes the form

$$C_2(SO(2,1)) = T_3(T_3 - 1) - T_+T_- = T_3(T_3 + 1) - T_-T_+ \tag{14}$$

and the operators satisfy the $SU(1, 1)$ algebra

$$[T_3, T_\pm] = \pm T_\pm \tag{15a}$$

$$[T_+, T_-] = -2T_3. \tag{15b}$$

Hence, $C_2(SO(2, 1)) \equiv C_2(SU(1, 1))$.

Assuming that the transition operators $T_\pm$ also act on the eigenstates $\psi_{v,l}$ or $\psi_{v,\lambda}$, Cooper [1] has shown that $T_\pm$ behave as ladder operators for the Coulomb and harmonic oscillator problems and as shift operators in the case of the Morse oscillator. This is sketched below very briefly.

Using (15*a*) and (12), it is straightforward to show that

$$T_3(T_\pm \psi_{v,l}^{\text{Coul}}) = (n \pm 1)(T_\pm \psi_{v,l}^{\text{Coul}}) \tag{16a}$$

$$T_3(T_\pm \psi_{v,l}^{\text{Harm}}) = (v + \tfrac{1}{2}l + \tfrac{3}{4} \pm 1)(T_\pm \psi_{v,l}^{\text{Harm}}) \tag{16b}$$

$$T_3(T_\pm \psi_{v,\lambda}^{\text{Morse}}) = (\lambda \pm 1)(T_\pm \psi_{v,\lambda}^{\text{Morse}}). \tag{16c}$$

Then, repeated use of (12) gives

$$T_\pm \psi_{v,l}^{\text{Coul}} \propto \psi_{v\pm1,l}^{\text{Coul}} = k_{\text{Coul}} \psi_{v\pm1,l}^{\text{Coul}} \tag{17a}$$

$$T_\pm \psi_{v,l}^{\text{Harm}} \propto \psi_{v\pm1,l}^{\text{Harm}} = k_{\text{Harm}} \psi_{v\pm1,l}^{\text{Harm}} \tag{17b}$$

$$T_\pm \psi_{v,\lambda}^{\text{Morse}} \propto \psi_{v\pm1,\lambda\pm1}^{\text{Morse}} = k_{\text{Morse}} \psi_{v\pm1,\lambda\pm1}^{\text{Morse}}. \tag{17c}$$

Notice that in the case of the Morse potential, the constant-energy condition demands that $(\lambda - v)$ remains constant, since $E_v = -(\lambda - v - \frac{1}{2})^2$.

The constant of proportionality k in each case follows from (15*b*) and (14), which allows us to write

$$T_\pm T_\mp = T_3(T_3 \mp 1) - C_2(SO(2, 1)). \tag{18}$$

Then, for each potential, using (11) and (12)

$$T_\pm T_\mp \psi_{v,l}^{\text{Coul}} = \{n(n \mp 1) - l(l + 1)\} \psi_{v,l}^{\text{Coul}} \tag{19a}$$

$$T_\pm T_\mp \psi_{v,l}^{\text{Harm}} = \{(v + \tfrac{1}{2}l + \tfrac{3}{4})(v + \tfrac{1}{2}l + \tfrac{3}{4} \mp 1) - \tfrac{1}{4}(l(l + 1) - \tfrac{3}{4})\} \psi_{v,l}^{\text{Harm}} \tag{19b}$$

$$T_\pm T_\mp \psi_{v,\lambda}^{\text{Morse}} = \{\lambda(\lambda \mp 1) - (\lambda - v)(\lambda - v - 1)\} \psi_{v,\lambda}^{\text{Morse}}. \tag{19c}$$

Choosing a phase factor of unity, equation (19) gives the constant of proportionality k in equation (17) as the square root of the curly bracket above, i.e. $k = \{\cdots\}^{1/2}$ for each case. Hence, we get the transition operators

$$T_\pm \psi_{v,l}^{\text{Coul}} = \{n(n \pm 1) - l(l + 1)\}^{1/2} \psi_{v\pm1,l}^{\text{Coul}} \tag{20a}$$

$$T_\pm \psi_{v,l}^{\text{Harm}} = \tfrac{1}{2}\{(2v + l + \tfrac{3}{2})(2v + l + \tfrac{3}{2} \pm 2) - l(l + 1) + \tfrac{3}{4}\}^{1/2} \psi_{v\pm1,l}^{\text{Harm}} \tag{20b}$$

$$T_\pm \psi_{v,\lambda}^{\text{Morse}} = \{\lambda(\lambda \pm 1) - (\lambda - v)(\lambda - v - 1)\}^{1/2} \psi_{v\pm1,\lambda\pm1}^{\text{Morse}}. \tag{20c}$$

Apparently, these transition operators represent ladder operators, i.e. the energy changing operators at constant angular momentum for the radial Coulomb and radial harmonic oscillator potentials, and as shift operators, i.e. changing the well-depth parameter but keeping the energy constant, for the Morse potential. Notice that in each case $T_- \psi_{0,l(\text{or }\lambda)} \equiv 0$, which defines the ground state of the system.

4. The bosonization procedure and the $SO_q(2,1)$ group

For the classical $SO(2,1)$ group, we have seen that its generators T_1, T_2, T_3 satisfy commutation relations (9) and transformations (13) lead to the $SU(1,1)$ algebra (15). The bosonic representation of $SU(1,1)$ in terms of two boson creation $(a_1^\dagger, a_2^\dagger)$ and two annihilation (a_1, a_2) operators can be defined as [20]

$$\begin{aligned} T_+ &= a_1^\dagger a_2^\dagger \\ T_- &= a_1 a_2 \\ T_3 &= \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) = \tfrac{1}{2}(N_1 + N_2 + 1) \end{aligned} \tag{21}$$

where $a_i^\dagger$, a_i $(i = 1, 2)$ satisfy the standard boson commutation rules. Then, for the representation $|N\omega\rangle$, the eigenvalue of the Casimir operator (10) or (14) of $SO(2,1)$ is [21]

$$C_2(SO(2,1))|N\omega\rangle = \tfrac{1}{4}\omega(\omega+2)|N\omega\rangle. \tag{22}$$

Here, N $(= N_1 + N_2)$ is the total number of bosons and ω is related [21] to the vibrational quantum number v:

$$v = \tfrac{1}{2}(N - \omega) \tag{23}$$

with $\omega = N, N-2, \ldots, 1$ or 0 (N = odd or even). Also, N is related to the maximum number of vibrational states:

$$N = 2v_{\max} \qquad \text{or} \qquad 2v_{\max} + 1 \qquad (N = \text{even or odd integer}). \tag{24}$$

For the q-deformation, using the same bosonic representation (21), but with

$$\begin{aligned} &a_i a_i^\dagger - q a_i^\dagger a_i = q^{-N_i} \qquad \text{or} \qquad a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i} \\ &[N_i, a_i^\dagger] = a_i^\dagger \\ &[N_i, a_i] = -a_i \end{aligned} \tag{25}$$

the commutation relations for $SU_q(1,1)$ become [18, 19]

$$\begin{aligned} &[T_3, T_\pm] = \pm T_\pm \\ &[T_+, T_-] = -[2T_3]. \end{aligned} \tag{26}$$

The first commutator in (26) can also be deformed, but this has been worked out so far only for the $SU_q(2)$ group [22]. Here, we have introduced a square bracket defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad (q = \mathrm{e}^s) \tag{27}$$

where s is real $(= a)$, imaginary $(= \mathrm{i}b)$ or, in general, complex $(= a + \mathrm{i}b)$. Notice that for $q \to 1$ (or $s \to 0$), $[x] \to x$.

The Fock space $\{|n\rangle_q\}$, $n = 0, 1, 2, \ldots$, for each oscillator is constructed as

$$\begin{aligned} &a|0\rangle_q = 0 \\ &|n\rangle_q = \frac{1}{\sqrt{[n]!}}(a^\dagger)^n|0\rangle_q \\ &[n]! = [n][n-1]\ldots[1]. \end{aligned} \tag{28a}$$

Also,

$$\begin{aligned} &a^\dagger|n\rangle_q = \sqrt{[n+1]}|n+1\rangle_q \qquad a|n\rangle_q = \sqrt{[n]}|n-1\rangle_q \\ &a^\dagger a = [N] \qquad aa^\dagger = [N+1] \qquad N|n\rangle_q = n|n\rangle_q. \end{aligned} \tag{28b}$$

Then, for two oscillators, letting $n_1 = \kappa$ and $n_2 = \mu$, the irreducible representation, built from (28), is the tensor product

$$|\kappa\mu\rangle_q = |n_1\rangle_q \otimes |n_2\rangle_q = \frac{1}{\sqrt{[\kappa]![\mu]!}}(a_1^\dagger)^\kappa (a_2^\dagger)^\mu |0\rangle_q. \tag{29}$$

Operating with T_3 and $T_\pm$ on equation (29), we obtain

$$\begin{aligned} &T_3|\kappa\mu\rangle_q = \mu|\kappa\mu\rangle_q \\ &T_\pm|\kappa\mu\rangle_q = \sqrt{[\mu \pm \kappa][\mu \mp \kappa \pm 1]}|\kappa\mu \pm 1\rangle_q \end{aligned} \tag{30}$$

with

$$\mu = \tfrac{1}{2}(n_1 + n_2 + 1) \tag{31a}$$

and

$$\kappa = \tfrac{1}{2}(1 + |n_1 - n_2|) \tag{31b}$$

where for any positive real number κ, $\mu = \kappa, \kappa + 1, \kappa + 2, \ldots$.

The second-order Casimir operator of $SU_q(1, 1)$ is [19]

$$C_2(SU_q(1,1)) = [T_3][T_3 - 1] - T_+T_- = [T_3][T_3 + 1] - T_-T_+ \tag{32}$$

whose eigenvalues for the above chosen representation are

$$C_2(SU_q(1,1))|\kappa\mu\rangle_q = [\kappa][\kappa - 1]|\kappa\mu\rangle_q. \tag{33}$$

Introducing the vibrational quantum number ω, defined as [11]

$$|n_1 - n_2| = \omega + 1 = 2\kappa - 1 \tag{34}$$

one obtains

$$C_2(SU_q(1,1))|\kappa\mu\rangle_q = \left[\frac{\omega}{2}\right]\left[\frac{\omega+2}{2}\right]|\kappa\mu\rangle_q. \tag{35}$$

We notice here that in the limit $q \to 1$ (or $s \to 0$), equation (35) reduces to equation (22), the Casimir of $SO(2, 1)$. Also, we have seen that $C_2(SO(2,1)) \equiv C_2(SU(1,1))$. Based on this result, which has been used previously in [12, 23], we find that the Casimirs of $SU_q(1, 1)$ and $SO_q(2, 1)$ are also the same, given by equation (32). Explicitly,

$$C_2(SO_q(2,1))(= C_2(SU_q(1,1))) = [T_3][T_3 - 1] - T_+T_- = [T_3][T_3 + 1] - T_-T_+. \tag{36}$$

5. q-deformed ladder/shift operators

For the q-deformation, the second commutator in (26), combined with equation (36), gives

$$T_{\pm}T_{\mp} = [T_3][T_3 \pm 1] \mp [2T_3] - C_2(SO_q(2,1)) \tag{37}$$

which, as in the classical case, has different eigenvalues for different potentials. Using the q-analogues of (11) and (12) in accordance with how (22) q-deforms to (35) and assuming that (37) act on the q-bosonic state $|\kappa\mu\rangle_q \equiv \psi^q$, we get for each potential

$$T_{\pm}T_{\mp}\psi^{q,\text{Coul}}_{v,l} = \{[n][n \pm 1] \mp [2n] - [l][l+1]\}\psi^{q,\text{Coul}}_{v,l} \tag{38a}$$

$$T_{\pm}T_{\mp}\psi^{q,\text{Harm}}_{v,l} = \{[v + \tfrac{1}{2}l + \tfrac{3}{4}][v + \tfrac{1}{2}l + \tfrac{3}{4} \pm 1] \mp [2v + l + \tfrac{3}{2}] - [\tfrac{1}{2}l - \tfrac{1}{4}][\tfrac{1}{2}l + \tfrac{3}{4}]\}\psi^{q,\text{Harm}}_{v,l} \tag{38b}$$

$$T_{\pm}T_{\mp}\psi^{q,\text{Morse}}_{v,\lambda} = \{[\lambda][\lambda \pm 1] \mp [2\lambda] - [\lambda - v][\lambda - v - 1]\}\psi^{q,\text{Morse}}_{v,\lambda}. \tag{38c}$$

Following the same phase convention as in the classical case,

$$T_{\pm}\psi^{q,\text{Coul}}_{v,l} = \{[n][n \mp 1] \pm [2n] - [l][l+1]\}^{1/2}\psi^{q,\text{Coul}}_{v\pm1,l} \tag{39a}$$

$$T_{\pm}\psi^{q,\text{Harm}}_{v,l} = \{[v + \tfrac{1}{2}l + \tfrac{3}{4}][v + \tfrac{1}{2}l + \tfrac{3}{4} \mp 1] \pm [2v + l + \tfrac{3}{2}] - [\tfrac{1}{2}1 - \tfrac{1}{4}][\tfrac{1}{2}l + \tfrac{3}{4}]\}^{1/2}\psi^{q,\text{Harm}}_{v\pm1,l} \tag{39b}$$

$$T_{\pm}\psi^{q,\text{Morse}}_{v,\lambda} = \{[\lambda][\lambda \mp 1] \pm [2\lambda] - [\lambda - v][\lambda - v - 1]\}^{1/2}\psi^{q,\text{Morse}}_{v\pm1,\lambda\pm1}. \tag{39c}$$

These are the q-deformed ladder and shift operators, respectively, for the radial Coulomb and radial harmonic oscillator potentials and the Morse potential (see, however, the discussion in the next section). For the ground state to be defined as in the classical case, we must also have

$$T_-\psi^q_{0,l(\text{or }\lambda)} = 0. \tag{40}$$

This condition is satisfied for all three problems since, from equation (27), we have the identity

$$[x][x+1] - [2x] - [x][x-1] \equiv 0 \tag{41}$$

for any arbitrary (real, imaginary or complex) value of q. Hence, equations (39) may be rewritten in the form

$$T_{\pm}\psi^{q,\text{Coul}}_{v,l} = \{[n][n \pm 1] - [l][l+1]\}^{1/2}\psi^{q,\text{Coul}}_{v\pm1,l} \tag{42a}$$

$$T_{\pm}\psi^{q,\text{Harm}}_{v,l} = \{[v + \tfrac{1}{2}l + \tfrac{3}{4}][v + \tfrac{1}{2}l + \tfrac{3}{4} \mp 1] - [\tfrac{1}{2}l - \tfrac{1}{4}][\tfrac{1}{2}l + \tfrac{3}{4}]\}^{1/2}\psi^{q,\text{Harm}}_{v\pm1,l} \tag{42b}$$

$$T_{\pm}\psi^{q,\text{Morse}}_{v,\lambda} = \{[\lambda][\lambda \pm 1] - [\lambda - v][\lambda - v - 1]\}^{1/2}\psi^{q,\text{Morse}}_{v\pm1,\lambda\pm1} \tag{42c}$$

explicitly demonstrating a one-to-one correspondence with classical operators (20) in the limit $q \to 1$, $[x] \to x$.

6. Summary and discussion of results

We have derived the quantum-deformed versions of the ladder and shift operators for three exactly solvable potentials: the radial Coulomb, radial harmonic oscillator and Morse oscillator potentials. Since the problems associated with these potentials correspond to different realizations of the $SO(2, 1)$ algebra, the quantum-deformed algebra of $SO(2, 1)$, namely $SO_q(2, 1)$, is established first. It is shown that the second-order Casimir operator of $SO_q(2, 1)$ is the same as that of $SU_q(1, 1)$.

Knowing that q-deformation involves sinh, sin or both sinh, sin and their cosine functions, the q-deformed operators have eigenvalues in far more complicated forms than their classical counterparts. Also, the quantum deformation parameter q varies continuously. This means that even for ladder operators with constant angular momentum l, the contribution of l to the energy shift could be varied by varying parameter q. Also, due to the sine function, there will be some degeneracies of states where the role of q on (constant) l-dependent terms will be important. In other words, the fine-structure effects are built into the q-formalism and the ladder operators for the Coulomb and harmonic oscillator potentials also seem to behave very much like shift operators, but due to changing q. Since the q-deformed shift operators for the Morse potential also depend on q, we refer to the q-deformed transition operators as *q-shift* operators. In this connection, it may be relevent to mention that an alternative algebraic approach of supersymmetric quantum mechanics [1], or the factorization method, also results in a set of shift operators only, for all three potentials studied here. For the radial harmonic oscillator problem, the shift operators act to change not only the l quantum number but also the energy. These are referred to as the quasi-shift operators [1]. Our q-shift operators for the quantum-deformation algebraic treatment of the three problems resemble the quasi-shift operators of the supersymmetric quantum mechanical treatment of the radial harmonic oscillator problem. Hence, a mapping between the two approaches may be possible.

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References

[1] Cooper I L 1993 *J. Phys. A: Math. Gen.* **26** 1601
[2] Kibler M and Négadi T 1991 *J. Phys. A: Math. Gen.* **24** 5283
[3] Gupta R K, Bach C T and Rosu H 1994 *J. Phys. A: Math. Gen.* **27** 1427
[4] Bonatsos D, Argyres E N, Drenska S B, Raychev P P, Roussev R P and Smirnov Yu F 1990 *Phys. Lett.* **251B** 447
[5] Gupta R K, Cseh J, Ludu A, Greiner W and Scheid W 1992 *J. Phys. G: Nucl. Part. Phys.* **18** L73
[6] Gupta R K 1994 *J. Phys. G: Nucl. Part. Phys.* **20** 1067
[7] Gupta R K and Ludu A 1993 *Phys. Rev.* C **48** 593
[8] Del Sol Mesa A, Loyola G, Moshinsky M and Velázquez V 1993 *J. Phys. A: Math. Gen.* **26** 1147
[9] Bonatsos D, Raychev P P, Roussev R P and Smirnov Yu F 1990 *Chem. Phys. Lett.* **175** 300
[10] Bonatsos D, Raychev P P and Faessler A 1991 *Chem. Phys. Lett.* **178** 221
[11] Bonatsos D, Argyres E N and Raychev P P 1991 *J. Phys. A: Math. Gen.* **24** L403

[12] Cao Y-F, Lin H-L 1993 *Chem. Phys. Lett.* **207** 317
[13] Kim S K, Cooper I L and Levine R D 1986 *Chem. Phys.* **106** 1
[14] Gupta R K and Cooper I L 1994 to be published
[15] Yang Q-G and Xu B-W 1993 *J. Phys. A: Math. Gen.* 26 L365
[16] Jagannathan R and Biswas S N 1993 *Preprint* MATSCIENCE 93/51
[17] Englefield M 1972 *Group Theory and the Coulomb Problem* (New York: Wiley)
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)
[18] Kulish P P and Damashinsky E V 1990 *J. Phys. A: Math. Gen.* 23 L415
[19] Ui H and Aizawa N 1990 *Mod. Phys. Lett.* A **5** 237
[20] Alhassid Y, Iachello F and Gürsey F 1983 *Chem. Phys. Lett.* **99** 27
[21] Lee D J, Shin K J and Kim S K 1990 *Chem. Phys. Lett.* **175** 87
[22] Ludu A and Gupta R K 1993 *J. Math. Phys.* **34** 5367
[23] Wang Y-C and Yang Z-S 1992 *Commun. Theor. Phys.* **17** 449